EXTREME AND MEAN RATIO IN VERGIL?

William C. Waterhouse

"I daresay you haven't had much practice," said the White Queen.

The basic thesis in G. E. Duckworth's *Structural Patterns and Proportions in Vergil's Aeneid* (Ann Arbor 1962) is that Vergil deliberately shaped his poetry to embody the extreme and mean ratio.[1] The same idea had been suggested earlier by G. LeGrelle,[2] and has been taken up by E. L. Brown.[3] Classicists have argued for and against the thesis,[4] but no one seems to have approached it from a background in mathematics. I have tried to consider the subject in the context of ancient mathematical thought, and the arguments presented below have forced me to conclude that Duckworth's thesis is extremely improbable. Most of the discussion will apply to all three authors, but I have concentrated on Duckworth's book because it is the most detailed presentation of the thesis (Brown devotes space to things like acrostic signatures,[5] and LeGrelle introduces a great deal of unrelated number symbolism[6]).

[1]A line is cut in extreme and mean ratio when, as the whole line is to the greater segment, so is the greater to the less (Euclid 6, Def. 3). In modern terms the ratio of the less to the greater is then $\frac{1}{2}(\sqrt{5} - 1)$, an irrational number whose decimal expansion begins 0.618 033 988 749 894 848. This ratio (or its inverse) has received from modern authors names like "divine proportion," "golden section," and "golden ratio." The ancients knew it only as "[section in] extreme and mean ratio."

[2]"Le premier livre des Géorgiques: poème pythagoricien," *LEC* 17 (1949) 139–235.

[3]*Numeri Vergiliani* (*Collection Latomus* 63, Brussels 1963).

[4]Favourable: G. B. Riddehough, *CJ* 58 (1963) 272–273; R. B. Lloyd, *AJP* 85 (1964) 71–77. Negative: A Dalzell, *Phoenix* 17 (1963) 314–316; R. D. Williams, *CP* 58 (1963) 248–251; M. L. Clarke, *CR* n.s. 14 (1964) 43–45.

[5]He has overlooked a striking Vergilian signature:

> . . . veniat fiducia, cunctas
> Posse capi: capies, tu modo tende plagas.
> VERe prius volucres taceant, aestate cicadae,
> MAenalius lepori dat sua terga canis . . . .

This surpasses Brown's acrostics, since it puts the three names in correct order and in successive lines. The only objection to it is that it was written by Ovid (*A.A.* 1.269–272).

[6]As an egregious example one might take "183, symbole des 183 univers pythagoriciens." LeGrelle (pp. 176–177) bases this interpretation on a quotation from Plutarch (*de defectu orac.* 422b). He does not point out that another character in the dialogue can without hesitation associate the theory of 183 worlds with a specific man: (422d) "His own book I have never read, nor am I sure that a copy still exists; but Hippys of Rhegium, mentioned by Phanias of Eresus, records that this opinion and account were Petron's . . . . He does not add anything to make it more plausible." Petron

Phoenix, Vol. 26 (1972) 4.

I. The fundamental fact is that ancient mathematicians divided all quantities into two essentially different types, discrete and continuous. Numbers were the mathematical representative of discrete quantity; continuous quantities were studied mathematically by geometry, and did not correspond to numbers. For a variety of mathematical and philosophical reasons numbers were thought to be more basic than continuous magnitudes. Now it was well known that extreme and mean ratio was irrational, arose in geometry, and could not be expressed by numbers. It is thus mathematically and philosophically unlikely that someone working with discrete numerical units would attempt to group them in an approximation of extreme and mean ratio.

The steps in this argument are probably unfamiliar outside history of mathematics, and need further explanation and justification. On the types of quantities, here is Aristotle: "Quantity is either discrete or continuous. . . . Instances of discrete quantities are number and speech; of continuous, lines, planes, solids, and also time and space."[7] Again, "that is called a multitude which is potentially divisible into non-continuous things; a magnitude, into continuous."[8] Eight centuries later Boethius is still drawing the same distinction:

Essentiae autem geminae partes sunt, una continua et suis partibus iuncta nec ullis finibus distributa, ut est arbor lapis et omnia mundi huius corpora, quae proprie magnitudines appellantur. Alia vero disiuncta a se et determinata partibus et quasi acervatim in unum redacta concilium, ut grex populus chorus acervus et quicquid, quorum partes propriis extremitatibus terminantur et ab alterius fine discretae sunt. His proprium nomen est multitudo.[9]

As Aristotle says, number (*numerus*, ἀριθμός) is an instance of discrete quantity. These words always refer to the counting numbers 2, 3, 4, . . . , and the corresponding verbs mean "to count." The ancients never conceived of a continuum of numbers. A number for Aristotle is "a combination of units,"[10] for Euclid "a multitude composed of units,"[11] for Nicomachus "a collection of units."[12] A fraction such as 3/4 was interpreted as meaning three units out of a total of four.

The fact is that these numbers are not capable of describing all the lines occurring in geometry. This fact was one of the major discoveries of early Greek mathematics, and naturally came as a shock. "All men are at first

is otherwise unknown, and only conjecturally a Pythagorean; there are no grounds whatever for attributing his "eccentric view" (Cornford, *CQ* 28 [1934] 14) to the entire Pythagorean school.

[7] *Cat.* 4b.

[8] *Metaph.* Δ, 1020a.

[9] *Inst. Arith.* 1.1. Cf. Nicomachus, *Arith.* 1.2.4.

[10] *Metaph.* Z., 1039a.

[11] 7, Def. 2.

[12] *Arith.* 1.7.1.

amazed by . . . the incommensurability of the diagonal of a square with its side. For it seems amazing, to all who have not studied the cause, that there is something which cannot be measured by any unit however small."[13] We have reacted to this discovery by broadening our concept of number; the ancients, however, concluded that the two kinds of quantities were essentially different. In later antiquity arithmetic and geometry are distinguished precisely by the types of quantities they consider: arithmetic deals with number, an abstract discrete quantity, while geometry considers continuous magnitudes like lines and angles.[14] The ingenious definition of Eudoxus[15] made it possible to talk about proportions involving magnitudes, but in that theory the "ratio" of two magnitudes is simply "the relation between them with respect to size"[16] and is not numerical. The proportions in Euclid 6 are statements not about length and area but about lines and rectangles.

Ancient authors who discuss the subject, especially those with Pythagorean leanings, have little doubt which branch of mathematics is more fundamental. "The one which naturally exists before them all, is superior, and holds the place of origin and root and, so to speak, mother of the others . . . is arithmetic."[17] Archytas says that arithmetic is clearer than geometry;[18] Aristotle, that it is more exact.[19] "It is clear to everyone," says Proclus, "that numbers are purer and more abstract than magnitudes."[20] (I hope the discussion has in fact made this clear: numbers were an abstract way of discussing discrete quantities, while individual magnitudes such as line segments had to be dealt with directly.) The skilled mathematician knew that numerical proportion was much simpler than the theory of proportion for magnitudes,[21] and even the beginner could appreciate Nicomachus's remark that you cannot define triangles without the concept of "three."[22] Behind all this there was a tradition attributing mystical and philosophical significance to numbers. The Pythagoreans were famous for believing that "things are numbers";[23] and the continuing importance of numbers in Plato, the Old Academy, Neopythagoreanism, and Neoplatonism is too well known to require discussion.

[13]Aristotle, *Metaph.* A, 983a.

[14]E.g. Nicomachus, *Arith.* 1.3.

[15]Euclid 5, Def. 5.

[16]Euclid 5, Def. 3.

[17]Nicomachus, *Arith.* 1.4.1.

[18]Diels, *Vorsokratiker*,[6] 47 B4.

[19]*Metaph.* A, 982a.

[20]*In Euc.* (ed. Friedlein) 95, 23.

[21]Faced with the problem of explaining what extreme and mean ratio is, Duckworth and the other authors all treat it in the modern way using numerical proportion (equality of fractions).

[22]*Arith.* 1.4.4.

[23]Aristotle, *Metaph.* M, 1083b.

When a line is cut in extreme and mean ratio, the pieces are incommensurable. In other words, the extreme and mean ratio is irrational: it cannot be expressed as a ratio of two whole numbers. It is unfortunate that Duckworth dismisses this fact and speaks of "the exact ratio .618."[24] A good deal of confusion seems to have been created, since several reviewers came away believing that 21/34 was extreme and mean ratio. Jackson Knight was led to say that Vergil "certainly followed, *often exactly* but sometimes approximately, the 'Golden Ratio' [italics mine]."[25] The fact therefore needs to be emphasized: *it is impossible to find two whole numbers in extreme and mean ratio.*

For modern men the mistake is perhaps easy to make, because we have a much broader concept of number; for us the extreme and mean ratio is in any case numerical, and whether it equals .618 exactly or only approximately seems to be a secondary question. But in antiquity the distinction was primary, and the place of extreme and mean ratio was clear: it belonged to geometry rather than arithmetic. "This cannot be expressed in terms of numbers," says the scholiast when it first occurs in Euclid.[26] Later in Euclid it is specifically classified as "the type of irrational line called apotome."[27] After listing irrational lines as one of the special topics in geometry, Proclus goes on to say "theorems about sections like those presented in Euclid 2 are common to both [sc. arithmetic and geometry], except for that in which the straight line is cut in extreme and mean ratio."[28]

By now the flaw in the following statement should be obvious: "If Vergil were a Neo-Pythagorean and as interested in the symbolic use of numbers as Maury and LeGrelle maintain, the Golden Section would certainly have appealed to him."[29] But Pythagoreanism is only a minor question. Extreme and mean ratio is a topic in theoretical geometry. (A practical handbook like Heron's *Metrica* does not discuss it, and it is never mentioned by the Roman *agrimensores*.) To learn about it in antiquity it was necessary to study pure mathematics. Anyone who did so would fully absorb the dichotomy between discrete and continuous quantities. It would then be thoroughly unnatural to think of arranging numbers in imperfect imitations of irrational straight lines. The idea would have no philosophical attraction, since numbers were if anything

[24]Duckworth, p. 39 and *passim.*

[25]*Roman Vergil* (Harmondsworth 1966), 425 = *PVS* 1 (1962) 5. LeGrelle actually makes an explicit statement (p. 142): "Il ne s'agit pas d'une application plus ou moins approchée de la *proportion divine*, mais de son exacte réalisation dans le compte des vers." I think it fair to say he had not fully grasped the point in question.

[26]*Ad* Euclid 2.11; ed. Heiberg, vol. 5, p. 248.

[27]Euclid 13.6.

[28]*In Euc.* (ed. Friedlein) 60, 16.

[29]Duckworth, p. 74.

purer than irrational lines. It is therefore extremely unlikely that Vergil or any other Roman poet attempted to approximate extreme and mean ratio in his verses.[30]

II. In addition to this fundamental flaw, there are several other parts of Duckworth's book which look weak in the light of ancient mathematics. For instance, he writes "I discovered that the Golden Mean ratios in many short passages were possible only if the half-lines were counted as fractions, i.e., .2, .3, .4, .6, .7, depending on the length of the half-line."[31] Now since a number was "a collection of units", anyone deliberately arranging objects in numerical patterns would naturally use multiples of a single basic unit. If Vergil were using the line as unit, he would not subdivide it; "good mathematicians, as you know, scornfully reject any attempt to cut up the unit itself into parts".[32] The other natural unit would be the foot, which is ruled out because the line is commonly divided at a caesura rather than a diaeresis. Duckworth at one point suggests the half-foot,[33] but that would not allow feminine caesuras. Besides, ancient authors do not mention it as a unit; feet were made of syllables, and syllables do not form just a numerical subdivision. Thus Duckworth has not actually made a case for any unit at all on which Vergil might have based his supposed numerical relations.

III. Another improbability is hidden in most of Duckworth's examples. By far the majority are formed by taking a set of verses divided into three or more parts and adding together some of the parts; in half the cases, the parts added are not contiguous. Now the section in extreme and mean ratio was just what its name implies (*sectio*, *τομή*): a way of cutting a line. A line segment in three pieces represents not one section but two. It is of course possible to imagine reshuffling and mentally rejoining the pieces to get a line cut in extreme and mean ratio, but this is a good deal more unnatural than it looks in Duckworth's purely numerical computations. The only place I have found such an idea is toward the end of a complicated proof in Euclid, and Heiberg there comments that the wording

[30]It is possible to doubt that extreme and mean ratio ever had aesthetic significance in antiquity; the role claimed for it in architecture is not mentioned by any ancient author. But in any case such a role would not affect the validity of the present argument, since dimensions of buildings are perceived as continuous quantities. (This would be true even if the architect used numerical approximations in the construction.) Lengths and the extreme and mean ratio are at least on the same side in the dichotomy between continuous and discrete; the point of my argument is that numbers are on the opposite side.

[31]Duckworth, p. 47. He later goes on to decide on this basis which of the lines were deliberately left incomplete.

[32]Plato, *Rep.* 525e.

[33]Duckworth, p. 66, n. 33.

"seems to indicate that the line is not in the true sense divided in extreme and mean ratio".[34]

IV. Still another point, one which requires a bit more discussion, is the supposed use of Fibonacci numbers.[35] Such a use is of course *a priori* unlikely, since these numbers first occur in Fibonacci's *Liber Abaci* of A.D. 1202. Duckworth is undaunted by this fact, believing that his own theory furnishes "conclusive proof that this series was known to the ancient Greek and Roman mathematicians".[36] But until the theory is better supported this hardly counts as evidence in its favor.

More discussion is required only because there are better reasons for supposing that the Fibonacci numbers might have been known earlier. Roughly, the argument is that the ancients knew something called "side and diagonal numbers" and that some of the ways these might have been derived would with slight modifications yield the Fibonacci numbers. At its best the argument becomes fairly attractive,[37] although of course it can never approach certainty. But fortunately, for our purposes the question need not be decided. In any such argument the side and diagonal numbers are early, in the period before irrationality was well understood. They have been preserved not in general mathematics books but rather in books for readers of Plato,[38] to explain the term "rational diameter" used in *Republic* 546c; and the authors say nothing of any related constructions. Thus even if the Fibonacci numbers were discussed in early Greece they clearly did not become part of the mainstream of mathematics. They might have been known to a few specialists later, but we can safely say that a Roman poet who studied a little mathematics[39] would never have heard of them.

[34]Euclid 13.17, ed. Heiberg, vol. 4, p. 327.

[35]The Fibonacci numbers are 1, 2, 3, 5, 8, . . . , each the sum of the previous two. The ratio of successive Fibonacci numbers approaches extreme and mean ratio.

[36]Duckworth, p. 63.

[37]See S. Heller, "Die Entdeckung der stetigen Teilung," in O. Becker (ed.), *Zur Geschichte der Griechischen Mathematik* (Darmstadt 1965), 319–354. Some of the relevant material is mentioned by Brown.

[38]Proclus, *In R.* (ed. Kroll) 2. 24–29; and Theo Smyrn. (ed. Hiller) 42–45. The discussion by Iamblichus (*In Nicom.*, ed. Pistelli, 91–93) seems to be taken from Theon; Nicomachus himself never mentions the subject.

[39]Duckworth might try to argue with this description, since he makes much of a sentence in the Suetonius-Donatus *Vita* of Vergil saying (15) "*inter cetera studia . . . maxime mathematicae operam dedit.*" Curiously, LeGrelle uses the same sentence to argue for Vergil's interest in astrology. (Suetonius in fact provides the Lewis-Short example of *mathematica* meaning astrology.) Donatus was generally inclined to endow Vergil with extraordinary wisdom—see D. Comparetti, *Vergil in the Middle Ages* (New York 1895), 56—and it would not take much study to surpass the usual Roman level of mathematical knowledge (cf. Cicero, *Tusc.* 1.5). At any rate the question is unimportant, since Duckworth goes on to find extreme and mean ratio in Lucretius and Horace.

Duckworth also believes Vergil used certain generalizations of the Fibonacci series, and Brown relies on one of them.[40] Here the question is much simpler, since the derivations suggested for side and diagonal numbers[41] do not yield these generalizations. Thus all the evidence allows us to say that no one in antiquity ever bothered with these numbers.

A few more words on Fibonacci numbers may be helpful, since not too many people are on familiar terms with them. If the ancients failed to discover them, it was not a serious failure; they are not of great importance in mathematics. In a technical sense they provide optimal approximations to extreme and mean ratio, although they are frequently not the best method for computation.[42] Fibonacci himself gives a thorough discussion of extreme and mean ratio without mentioning the Fibonacci numbers;[43] when he does introduce them they carry no suggestion of symbolic or aesthetic value, being related to the reproduction of rabbits.[44]

V. Duckworth has tabulated a large number of passages to support his thesis. If one wanted to analyse his data in detail, the first step would necessarily be to judge whether the *Aeneid* excerpts are fairly chosen and properly subdivided; for this I have no special qualifications.[45] But even accepting the tables as given one can perform a few quick tests which indicate that the specific value of extreme and mean ratio plays no exceptional role. Duckworth's tables consist of fractions with values between .600 and .639; he computes the values to three decimal places. For comparison, I listed all the possible fractions in this range with denominator less than 70. Duckworth finds that 4% of his entries give .618, the first three digits of extreme and mean ratio; and I find that 3% of the fractions chosen at random give that value. The comparison suggests that his fractions actually may not be too far from random.[46]

[40]The generalizations are formed by taking two numbers at random and then continuing with each term the sum of the previous two. The numbers suggested by Brown, for example, are 2, 8, 10, 18, 28, 46, ...

[41]See Heller, *op. cit.*, and B. L. Van der Waerden, *Science Awakening* (New York 1961), 127.

[42]It is for instance easier to compute iterated approximations to $\sqrt{5}$. Starting with the rough value 9/4 and applying the process three times gives the approximation 5, 374, 978, 561/2, 403, 763, 488, good enough to yield the eighteen decimals given in note 1. Unlike Fibonacci numbers, the iteration process is attested in antiquity and was probably known to the Babylonians—see O. Becker, *Das mathematische Denken der Antike* (Göttingen 1957), 64

[43]Leonardo of Pisa (Fibonacci), ed. Boncompagni, 2, 196–197.

[44]Ibid. 1, 283–284: *Quot paria coniculorum in uno anno ex uno pario germinentur.*

[45]Duckworth's subdivisions have been questioned by several reviewers, and are attacked by J. P. Bews, "*Aeneid* 1 and .618?" *Phoenix* 24 (1970) 130–143.

[46]On p. 46 Duckworth argues that this cannot be true, but his argument is hardly persuasive.

I also checked the numbers .600, .610, .620, and .630, since it was not hard to spot them in the tables. Of them, .620 and .610 are the closest to extreme and mean ratio, followed by .630, with .600 the furthest away. If the numbers were deliberately concentrated around extreme and mean ratio, we would thus expect .620 to be the most common and .600 the least. In fact, it is quite the opposite: .620 and .610 occur 22 and 21 times, respectively, while .630 occurs 28 times and .600 no less than 95 times. This agrees well with the fact that on my random list .600 is far more common than the others. Once again the tables do not seem to distinguish extreme and mean ratio from other ratios in the same range.

Similarly, what Duckworth calls (p. 47) "the overwhelming preference for the Fibonacci series" turns out to be largely imaginary. He reports (p. 63) that 47% of his integral ratios are Fibonacci ratios, and on my random list the Fibonacci ratios are 31% of the total and 58% of those with denominator less than 30.

As a final test I took one of the less trivial Fibonacci ratios, that where the larger part is 34 and the smaller is 21. This gives a relatively good approximation to extreme and mean ratio, and Duckworth points out on p. 61 that it occurs twice in his tables. A search through the tables confirmed this but also showed that 33:19 occurs twice, 33:20 occurs six times, 33:21 occurs three times, 34:22 occurs three times, 35:22 occurs four times, and 35:23 occurs twice. All of these are significantly worse approximations, some of them quite bad; and every one of them could have been changed to 34:21 by writing a couple of lines more or less. It is particularly striking that 34:21 actually occurs less often than 33:21 and 34:22. Clearly the "good" ratio 34:21 has no special role.

In summary, Duckworth's data seem to show no emphasis on Fibonacci numbers, no substantial clustering at .618, and no preference for good approximations over bad ones; they behave more like a random collection of fractions in the range considered. Thus they do not contradict the conclusion drawn in the rest of this paper: there is no reason to believe that Vergil used extreme and mean ratio, and good reason to suppose that he did not.

Department of Mathematics, Cornell University